Math 255B Lecture 25 Notes

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March 9, 2020

1 Projection-Valued Measures

1.1 Unbounded functional calculus

Let A be self-adjoint, and let $\varphi \in \mathrm{Ba}(\mathbb{R})$. We want to define $\varphi(A)$ by

$$D(\varphi(A)) = \left\{ u \in H : \int |\varphi(\lambda)|^2 \, d\mu_u(\lambda) < \infty \right\},$$

$$\varphi(A)u = \lim_{j \to \infty} \varphi_j(A)u, \qquad \varphi_j \in \mathrm{Ba}_b(\mathbb{R}), \varphi_j \to \varphi, |\varphi_j| \leq |\varphi| < \infty.$$

Proposition 1.1. *Let A be self-adjoint, and let $\varphi \in \mathrm{Ba}(\mathbb{R})$. Then $\varphi(A)$ is densely defined, and $\varphi(A)^* = \overline{\varphi}(A)$ (so $D(\varphi(A)^*) = D(\overline{\varphi})$). In particular, $\varphi(A)$ is closed.*

Proof. Let's check the first claim. If $u \in H$, let $\chi_n = \mathbb{1}_{\{|\varphi| \leq n\}}$. Then $\chi_n \in \mathrm{Ba}_b$, and $\chi_n \to 1$. So $\chi_n(A)u \to u$. Notice that if $u \in H$ and $\varphi \in \mathrm{Ba}_b(\mathbb{R})$, then $d\mu_{\varphi(A)u} = |\varphi|^2 \, d\mu_u$: for any $f \in C_0$,

$$\int f \, d\mu_{\varphi(A)u} = \langle f(A)\varphi(A)u, \varphi(A)u \rangle = \left\langle (f|\varphi|^2)(A)u, u \right\rangle = \int f|\varphi|^2 \, d\mu_u.$$

In particular, $\mathrm{supp}(d\mu_{\chi_n(A)u}) \subseteq \{|\varphi| \leq n\}$, so

$$\int |\varphi|^2 \, d\mu_{\chi_n(A)u} < \infty,$$

which implies that $\chi_n(A)u \in D(\varphi(A))$. Now use this argument with $\chi_n \varphi$ to get the claim. $\square$

1.2 Projection-valued measures

Let $M \subseteq \mathbb{R}$ be a Baire set ($\mathbb{1}_M \in \mathrm{Ba}$) and set $E(M) := \mathbb{1}_M(A) \in \mathcal{L}(H, H)$. Then

- $E(M)$ is an orthogonal projection on H.

- $E(\mathbb{R}) = 1$.
- $E(M_1)E(M_2) = E(M_1 \cap M_2)$.
- $E(\varnothing) = 0$.
- If $M = \bigcup_{j=1}^{\infty} M_j$ with $M_j \cap M_k = \varnothing$, then $E(M)$ is the strong limit of $\sum_{j=1}^{N} E(M_j)$ as $N \to \infty$.

The map $M \mapsto E(M)$ is an **orthogonal projection-valued measure**.

Recall next that if μ is a positive bounded measure on $\mathbb{R}$, then there exists a unique left continuous increasing function φ such that $\varphi(\lambda) \to 0$ as $\lambda \to -\infty$ and $\int f\, d\mu = \int f\, d\varphi$ for all $f \in C_0$: We have $\varphi(\lambda) = \mu(\mathbb{1}_{(-\infty,\lambda)})$.

In particular,

$$\mu_u(\mathbb{1}_{(-\infty,\lambda)}) = \left\langle \mathbb{1}_{(-\infty,\lambda)}(A)u, u \right\rangle = \langle E_\lambda u, u\rangle\,, \qquad \text{where } E_\lambda = E((-\infty,\lambda)) = \mathbb{1}_{(-\infty,\lambda)}(A).$$

We have:

- $E_\lambda u \to 0$ as $\lambda \to -\infty$,
- $E_\lambda u \to u$ as $\lambda \to \infty$,
- $E_{\lambda-\varepsilon} u \to E_\lambda u$ as $\varepsilon \to 0^+$.

If $\varphi \in \mathrm{Ba}_b$, we have

$$\langle \varphi(A)u, u\rangle = \int \varphi(\lambda)\, d\mu_u(\lambda) = \int \varphi(\lambda)\, d\langle E_\lambda u, u\rangle.$$

If $u \in D(A)$, then $Au = \lim_{j\to\infty} A\varphi_j(A)u$, where $0 \le \varphi_j \le 1$ with $\varphi_j \in \mathrm{Ba}$ and $\mathrm{supp}(\varphi_j)$ compact. Then

$$\langle Au, u\rangle = \lim_{j\to\infty} \int \lambda\varphi_j(\lambda)\, d\langle E_\lambda u, u\rangle = \int \lambda\, d\langle E_\lambda u, u\rangle.$$

Formally, we write

$$A = \int \lambda\, dE_\lambda.$$

1.3 Properties of projection-valued measures

There is nothing new here, but we are taking a different (and useful) point of view. Here is a proposition that expresses this point of view.

Proposition 1.2.

1. *Let $\lambda \in \mathbb{R}$. Then $\lambda \in \operatorname{Spec}(A)$ if and only if $E((\lambda-\varepsilon, \lambda+\varepsilon)) \neq 0$ for all $\varepsilon > 0$.*
2. *$E(\{\lambda\}) \neq 0$ if and only if λ is an eigenvalue of A and $E(\{\lambda\})$ is the orthogonal projection onto $\ker(A-\lambda)$.*

Proof.

1. ($\Longleftarrow$): For any $\varepsilon > 0$, pick $u_\varepsilon \in H$ such that $\|u_\varepsilon\| = 1$ and $u_\varepsilon = E((\lambda-\varepsilon, \lambda+\varepsilon))u_\varepsilon$. Then $d\mu_{u_\varepsilon}(t) = \mathbb{1}_{(\lambda-\varepsilon,\lambda+\varepsilon)}\, d\mu_{u_\varepsilon}(t)$. So $u_\varepsilon \in D(A)$ and

$$\|(A-\lambda)u_\varepsilon\| = \int (t-\lambda)^2\, d\mu_{u_\varepsilon}(t) \leq \varepsilon^2 \int d\mu_{u_\varepsilon}(t) = \varepsilon^2$$

It follows that $\lambda \in \operatorname{Spec}(A)$: for any $\varepsilon > 0$ there is a $u_\varepsilon \in H$ such that $\|u_\varepsilon\| = 1$ and $\|(A-\lambda)u_\varepsilon\| \leq \varepsilon$, so $A - \lambda$ has no bounded inverse.

($\Longrightarrow$): If $E((\lambda-\varepsilon, \lambda+\varepsilon)) := E(I_\varepsilon) = 0$ for some $\varepsilon > 0$, then for any $u \in H$,

$$u = E(\mathbb{R})u = E(\mathbb{R} \setminus I_\varepsilon)u$$

Then for any $u \in H$, $\operatorname{dist}(\lambda, \operatorname{supp}(\mu_u)) \geq \varepsilon$. Let $\varphi(t) = \frac{1}{t-\lambda}$, which is continuous and bounded on $\operatorname{supp}(\mu_u)$ for all u. Then define $\varphi(A) \in \mathcal{L}(H,H)$ by $\langle \varphi(A)u, u\rangle = \int \varphi(t)\, d\mu_u(t)$. We get $(A-\lambda)\varphi((A) = 1$, so $\lambda \notin \operatorname{Spec}(A)$.

2. ($\Longrightarrow$): If $E(\{\lambda_0\}) \neq 0$, then let $u \neq 0$ be such that $u = E(\{\lambda_0\})u$. Then $\operatorname{supp}(\mu_u(\lambda)) \subseteq \{\lambda_0\}$, so $u \in D(A)$ and

$$\|(A-\lambda_0)u\|^2 = \int (\lambda-\lambda_0)^2\, d\mu_u(\lambda) = 0.$$

So λ_0 is an eigenvalue, and $\operatorname{im} E(\{\lambda_0\}) \subseteq \ker(A - \lambda - 0)$.

On the other hand, if $(A-\lambda_0)u = 0$ for some $0 \neq u \in D(A)$, then for $\operatorname{Im} z \neq 0$, $(A-z)u = (\lambda_0 - z)u$, so

$$R(z)u = \frac{1}{\lambda_0 - z}u, \qquad \langle R(z)u, u\rangle = \frac{\|u\|^2}{\lambda_0 - z}.$$

So by the uniqueness in our Nevanlinna representation, we get $\mu_u(\lambda) = \|u\|^2\delta(\lambda-\lambda_0)$. So for any $\varphi \in \mathrm{Ba}_b$, $\varphi(A)u = \varphi(\lambda_0)u$. So $E(\{\lambda_0\}) \neq 0$, and $E(\{\lambda_0\})u = u$. Thus, $\operatorname{im} E(\{\lambda_0\}) = \ker(A-\lambda_0)$. $\square$